Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons.

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SECOND-SOUND PHENOMENA IN INVISCID, THERMALLY RELAXING GASES

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Abstract. We consider the propagation of acoustic and thermal waves in a class of inviscid, thermally relaxing gases wherein the flow of heat is described by the Maxwell–Cattaneo law, i.e., in Cattaneo–Christov gases. After first considering the start-up piston problem under the linear theory, we then investigate traveling wave phenomena under the weakly-nonlinear approximation. In particular, a shock analysis is carried out, comparisons with predictions from classical gases dynamics theory are performed, and critical values of the parameters are derived. Special case results are also presented and connections to other fields are noted.

1. Introduction.

Setting aside the fact that it has been shown to fail[1] in situations involving low temperature and/or high heat flux conditions, the constitutive relation known as Fourier's law, which for present purposes may be expressed as

$$\mathbf{q} = -K\nabla\vartheta, \tag{1}$$

where $\vartheta(>0)$, $\mathbf{q}$, and $K(>0)$ denote the absolute temperature, heat flux vector, and thermal conductivity, respectively, suffers from a fundamental drawback: What has come to be known as the *paradox of heat conduction* (PHC) [17, 36]. This refers to the fact that implicit in the use of Eq. (1) is the assumption that, in a continuous medium, a thermal disturbance at one point will be felt instantly, but unequally, at all others, however distant. Such behavior is, of course, physically unrealistic and constitutes a violation of the well-established principle of classical mechanics known as *causality* [47].

Enter now the Maxwell–Cattaneo (MC) flux law, which in the case of a rigid, isotropic solid at rest assumes its simplest form

$$\tau\mathbf{q}_t + \mathbf{q} = -K\nabla\vartheta, \tag{2}$$

where $\tau(>0)$ has been termed the *intrinsic relaxation time of the heat flux* [13]. Under this constitutive relation, the flow of heat within a continuous medium does not occur instantaneously; rather, it does so over time via the propagation of thermal waves, a phenomenon known as *second-sound* [17, 24, 36, 45]. The basis of Eq. (2) can be traced back to Maxwell's work on the kinetic theory of gases, specifically, his seminal 1867 paper entitled "On the dynamical theory of gases" [30].

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[1] See, e.g., Chandrasekharaiah [8, p. 357] and Dreyer and Struchtrup [17, p. 4].

However, it was not until the 1950s, following the detection of second-sound in He II by Peshkov [37] and the formulation of a simple hyperbolic theory to describe heat conduction in gases by Cattaneo [7], that widespread interest in thermal wave phenomena began to develop; see, e.g., Refs. [8, 45, 50] and those therein.

As the title indicates, this communication is devoted to the study of second-sound in gases. Unlike those carried out by Maxwell and Cattaneo, however, the present investigation is performed under the framework of continuum mechanics, and as such can be regarded as complementing the works of Carrassi and Morro [5], Lindsay and Straughan [28], and Straughan [44] on second-sound in fluids, as well as those of Moran and Shen [32] and Lesser and Seebass [27] from the field of classical acoustics. Moreover, while Grad [20] and others[2] have derived sophisticated generalizations of Eq. (2) to describe the flow of heat in virtually all forms of continuous media, here, following Straughan [44] and Tibullo and Zampoli [48], we take as our flux relation the generalization of the MC law put forth by Christov [10], namely,

$$\tau\left[\mathbf{q}_t + (\mathbf{u}\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\mathbf{u} + (\nabla\cdot\mathbf{u})\mathbf{q}\right] + \mathbf{q} = -K\nabla\vartheta, \tag{3}$$

which in terms of generality will prove more than adequate for the analysis carried out below. Here, $\mathbf{u}$ denotes the velocity vector (see section 2) and we observe that, while not relevant to the present study, Eq. (3) possesses the property of frame-indifference, which Christov achieved via the use of a Lie–Oldroyd derivative, instead of $\partial/\partial t$, to describe the time-rate-of-change of $\mathbf{q}$; see also Refs. [26, 44].

Our primary aim here is to investigate the propagation of acoustic and second-sound waves in what Straughan [44] has termed a "Cattaneo–Christov" gas, i.e., an inviscid perfect gas that, along with the ability to conduct heat, exhibits thermal relaxation described by Eq. (3). In particular, our focus shall be on shock and traveling wave phenomena, in one-dimension (1D), under both the linear and weakly-nonlinear formulations of the system of governing equations. The present investigation, therefore, seeks to examine phenomena not considered by Straughan [44], whose focus was limited to acceleration waves, nor by Tibullo and Zampoli [48], whose uniqueness result was based on the incompressibility assumption.

To this end, the exposition contained herein is organized as follows. In section 2, the governing equations and constitutive relations are developed. Next, in section 3, shocks admitted under the linear theory are examined in the context of a classic signaling problem from acoustics. Then, in section 4, traveling wave and shock phenomena are analyzed under the weakly-nonlinear approximation. And finally, in section 5, final remarks are made and possible extensions of the present study are suggested.

Remark 1. While neglecting viscosity but not heat conduction is, of course, a theoretical idealization [41, p. 179], particularly in the case of gases, Straughan's [44] Cattaneo–Christov model possesses two features that make it highly desirable vis-à-vis the present study; it is (strictly) hyperbolic, thus ensuring the requirements of causality are satisfied [40, p. 5], and it allows the effects of thermal relaxation to be studied without obfuscation from those of viscosity.

2. **Balance laws and constitutive assumptions.** Consider a compressible inviscid gas, wherein we assume the flow of heat is described by Eq. (3) with $\tau(= \tau_0)$

[2]See, in particular, the contributions of Morro [33], Müller [35], and Ruggeri [39]; see also Refs. [24, 36, 45] and those therein.

and $K(=K_0)$ both constant, that obeys the equation of state [29, Eq. (1.1)]

$$\wp = \wp_0(1+s)^\gamma \exp[(\eta-\eta_0)/c_v]. \tag{4}$$

Equation (4), it should be noted, is derived from the thermodynamic axiom

$$\vartheta\, \mathrm{d}\eta = c_v\, \mathrm{d}\vartheta - \wp\varrho^{-2}\mathrm{d}\varrho, \tag{5}$$

known as the *Gibbs equation* [46], with aid of the perfect gas law, namely, $\wp = c_v(\gamma-1)\varrho\vartheta$, which we observe can also be expressed as

$$\wp = \wp_0(1+s)(1+\theta). \tag{6}$$

Here, $\wp(>0)$ is the thermodynamic pressure; η denotes the specific entropy; $s = (\varrho-\varrho_0)/\varrho_0$ is known as the condensation, where $\varrho(>0)$ is the mass density; $\theta = (\vartheta-\vartheta_0)/\vartheta_0$; the exponent $\gamma = c_p/c_v$ is known as the adiabatic index, where the constants $c_p > c_v > 0$ denote the specific heats at constant pressure and volume, respectively, and $1 < \gamma \leq 5/3$; and $\wp_0$, ϱ_0, ϑ_0, and η_0 are constants that denote the equilibrium state values of the corresponding thermodynamic variables, where by equilibrium state we mean the unperturbed state characterized by $u = 0$, $\wp = \wp_0$, $\varrho = \varrho_0$, $\vartheta = \vartheta_0$, and $\eta = \eta_0$.

In the case of such a gas, the continuity, momentum, and entropy equations can be expressed as

$$\dot{s} + (\nabla\cdot\mathbf{u})(1+s) = 0, \tag{7}$$

$$\varrho_0(1+s)[\mathbf{u}_t + \tfrac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times(\nabla\times\mathbf{u})] = -\nabla\wp, \tag{8}$$

$$\varrho_0\vartheta_0(1+s)(1+\theta)\dot{\eta} = -\nabla\cdot\mathbf{q}, \tag{9}$$

where the absence of both external body forces and internal heat sources has been assumed and a superposed dot represents the material derivative.

Henceforth restricting our attention to plane wave propagation along the x-axis, it follows that $\mathbf{u} = (u(x,t),0,0)$ and $\mathbf{q} = (q(x,t),0,0)$, while $\wp$, ϱ, ϑ, and η all become functions of x and t only. As a result, Eqs. (7)–(9) and (3) are reduced to

$$s_t + us_x + (1+s)u_x = 0, \tag{10}$$

$$\varrho_0(1+s)(u_t + uu_x) = -\wp_x, \tag{11}$$

$$\varrho_0\vartheta_0(1+s)(1+\theta)(\eta_t + u\eta_x) = -q_x, \tag{12}$$

$$q + \tau_0(q_t + uq_x) = -K_0\vartheta_0\theta_x, \tag{13}$$

respectively.

In concluding this section, we call attention to the fact that η can be eliminated from Eq. (12) using Eq. (5); the former can thus be recast, after eliminating $\wp$ via the perfect gas law, as

$$c_v\varrho_0\vartheta_0[(1+s)(\theta_t + u\theta_x) - (\gamma-1)(1+\theta)(s_t + us_x)] = -q_x, \tag{14}$$

a form of the internal energy equation which we shall make use of in the next section.

3. **Linear theory: The start-up piston problem.** In this section we consider a semi-infinite pipe, the central axis of which coincides with the $+x$-axis, that is filled with an inviscid perfect gas, within which the flow of heat is described by the MC law. Adopting the flow geometry of Ref. [32], i.e., planar propagation along the $+x$-axis, and linearizing[3], Eq. (10), (11), (14), and (6) simplify to

$$s_t = -u_x, \quad \varrho_0 u_t = -\wp_x, \quad c_v \varrho_0 \vartheta_0[\theta_t - (\gamma-1)s_t] = -q_x, \quad \wp = \wp_0(1+s+\theta), \tag{15}$$

respectively, while Eq. (13) assumes the usual form of the MC law, namely,

$$q + \tau_0 q_t = -K_0 \vartheta_0 \theta_x. \tag{16}$$

Eliminating $\wp$ between Eqs. $(15)_{2,4}$ and q between Eqs. $(15)_3$ and (16), the momentum and energy equations become

$$u_t = -b^2(s_x + \theta_x), \quad (1+\tau_0\partial_t)[\theta_t - (\gamma-1)s_t] = \gamma\kappa\theta_{xx}, \tag{17}$$

where $b = \sqrt{\wp_0/\varrho_0}$ is known as the *isothermal* sound speed and $\kappa = K_0/(\varrho_0 c_p)$ denotes the thermal diffusivity. Now using Eq. $(15)_1$ to eliminate s from both Eqs. $(17)_{1,2}$, our system assumes it final form

$$u_{tt} - b^2 u_{xx} = -b^2\theta_{tx}, \qquad \theta_t + \tau_0\theta_{tt} - \gamma\kappa\theta_{xx} = -(\gamma-1)(1+\tau_0\partial_t)u_x. \tag{18}$$

As did Moran and Shen [32], we assume the following: (i) initially, the gas is in its equilibrium state; (ii) at time $t = 0^+$ the piston begins advancing to the right, from the plane $x = 0$, with constant speed, which we denote here by $u_0(>0)$; and (iii), the face of the piston is thermally insulated. Mathematically, these physical assumptions translate into the following boundary and initial conditions:

$$u(0,t) = u_0 H(t), \quad u(\infty,t) = 0, \quad \theta_x(0,t) = 0, \quad \theta(\infty,t) = 0 \quad (t>0); \tag{19}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad \theta(x,0) = 0, \quad \theta_t(x,0) = 0 \quad (x>0), \tag{20}$$

respectively, where $H(\cdot)$ denotes the Heaviside unit step function.

Applying $\mathcal{L}[\,\cdot\,]$, the Laplace transform [6, 18] with respect to t, to System (18) and then making use of the initial conditions (IC)s yields, after simplifying, the system of subsidiary equations

$$b^2\bar{u}_{xx} - \alpha^2\bar{u} = b^2\alpha\bar{\theta}_x, \qquad \gamma\kappa\bar{\theta}_{xx} - (\alpha+\tau_0\alpha^2)\bar{\theta} = (\gamma-1)(1+\tau_0\alpha)\bar{u}_x. \tag{21}$$

Here, α denotes the Laplace transform parameter; a bar over a quantity denotes the image of that quantity in the Laplace transform domain, e.g., $\bar{\theta}(x,\alpha) := \mathcal{L}[\theta(x,t)]$; and we note for future reference that

$$\alpha\bar{s} = -\bar{u}_x, \qquad \bar{\wp} = \wp_0(\bar{s} + \bar{\theta} + 1/\alpha), \qquad (1+\tau_0\alpha)\bar{q} = -K_0\vartheta_0\bar{\theta}_x, \tag{22}$$

where $q(x,0) = 0$ was assumed[4] in deriving Eq. $(22)_3$.

Solving System (21) subject to the transformed set of boundary conditions, i.e.,

$$\bar{u}(0,\alpha) = u_0/\alpha, \quad \bar{u}(\infty,\alpha) = 0, \quad \bar{\theta}_x(0,\alpha) = 0, \quad \bar{\theta}(\infty,\alpha) = 0, \tag{23}$$

it is readily established that

$$\bar{\theta}(x,\alpha) = -\frac{u_0(\gamma-1)(\alpha+\tau_0\alpha^2)}{\kappa c_0^2[m_2^2(\alpha) - m_1^2(\alpha)]}\left\{\frac{\exp[-m_2(\alpha)x]}{m_2(\alpha)} - \frac{\exp[-m_1(\alpha)x]}{m_1(\alpha)}\right\}, \tag{24}$$

[3] It should be noted that Carrassi and Morro [5] also derive a linearized system of flow equations based on the MC law.

[4] From the physical standpoint, imposing $q(x,0) = 0$ is obvious; mathematically, however, $\theta_t(x,0) = 0$, the corresponding temperature IC (see Eq. $(20)_4$), implies only that $q(x,0) \equiv$ const.

$$\bar{u}(x,\alpha) = \frac{u_0\gamma\alpha}{c_0^2}\left\{\frac{\exp[-m_2(\alpha)x] - \exp[-m_1(\alpha)x]}{m_2^2(\alpha) - m_1^2(\alpha)}\right\} - \frac{u_0(\alpha^2+\tau_0\alpha^3)}{\kappa c_0^2[m_2^2(\alpha) - m_1^2(\alpha)]}\left\{\frac{\exp[-m_2(\alpha)x]}{m_2^2(\alpha)} - \frac{\exp[-m_1(\alpha)x]}{m_1^2(\alpha)}\right\}, \quad (25)$$

where

$$m_{1,2}(\alpha) = \sqrt{\frac{\alpha}{2\kappa}} \times \sqrt{1+(\tau_0+\gamma\kappa c_0^{-2})\alpha \pm \sqrt{[1+(\tau_0+\gamma\kappa c_0^{-2})\alpha]^2 - 4\kappa c_0^{-2}(\alpha+\tau_0\alpha^2)}}\,. \quad (26)$$

From the above expressions for $\bar{\theta}$ and $\bar{u}$, the barred version of each of the remaining field variables can be determined via Eqs. (22); e.g., from Eq. $(22)_3$ we get

$$\bar{q}(x,\alpha) = -\frac{u_0K_0\vartheta_0(\gamma-1)\alpha}{\kappa c_0^2}\left\{\frac{\exp[-m_2(\alpha)x] - \exp[-m_1(\alpha)x]}{m_2^2(\alpha) - m_1^2(\alpha)}\right\}. \quad (27)$$

3.1. **Shock amplitude expressions: Derivation.** We begin by expanding the right-hand side of Eq. (24) for large-α, the result of which we write as

$$\bar{\theta}(x,\alpha) = \exp(-\sigma_2 x)\left[\frac{\mathcal{A}_2}{\alpha} + \mathcal{O}(\alpha^{-2})\right]\exp\left[-\frac{\alpha x}{v_2}\left(1+\mathcal{O}(\alpha^{-2})\right)\right] + \exp(-\sigma_1 x)\left[\frac{\mathcal{A}_1}{\alpha} + \mathcal{O}(\alpha^{-2})\right]\exp\left[-\frac{\alpha x}{v_1}\left(1+\mathcal{O}(\alpha^{-2})\right)\right]. \quad (28)$$

Here, and throughout this subsection, $\tau_0 > 0$ is assumed;

$$\mathcal{A}_1 = -\frac{u_0c_0\tau_0(\gamma-1)\sqrt{2\kappa}}{\sqrt{\Pi(\tau_0)\left[\gamma\kappa + c_0^2\tau_0 + \sqrt{\Pi(\tau_0)}\,\right]}}, \quad (29)$$

$$v_1 = c_0\sqrt{\frac{2\kappa}{\gamma\kappa + c_0^2\tau_0 + \sqrt{\Pi(\tau_0)}}}\,, \quad (30)$$

$$\sigma_1 = \frac{c_0}{2\sqrt{2\kappa}}\left\{\frac{c_0^2\tau_0 - \kappa(2-\gamma) + \sqrt{\Pi(\tau_0)}}{\sqrt{\Pi(\tau_0)\left[\gamma\kappa + c_0^2\tau_0 + \sqrt{\Pi(\tau_0)}\,\right]}}\right\}; \quad (31)$$

$$\mathcal{A}_2 = \frac{u_0c_0\tau_0(\gamma-1)\sqrt{2\kappa}}{\sqrt{\Pi(\tau_0)\left[\gamma\kappa + c_0^2\tau_0 - \sqrt{\Pi(\tau_0)}\,\right]}}, \quad (32)$$

$$v_2 = \sqrt{\frac{\gamma\kappa + c_0^2\tau_0 + \sqrt{\Pi(\tau_0)}}{2\tau_0}}\,, \quad (33)$$

$$\sigma_2 = \frac{c_0}{2\sqrt{2\kappa}}\left\{\frac{\kappa(2-\gamma) - c_0^2\tau_0 + \sqrt{\Pi(\tau_0)}}{\sqrt{\Pi(\tau_0)\left[\gamma\kappa + c_0^2\tau_0 - \sqrt{\Pi(\tau_0)}\,\right]}}\right\}; \quad (34)$$

and we have set $\Pi(\tau_0) := (\gamma\kappa)^2 - 2\kappa c_0^2(2-\gamma)\tau_0 + c_0^4\tau_0^2$ for convenience, where we observe that $\Pi(\tau_0) > 0$ since $\gamma > 1$.

On applying the theorem of Boley and Hetnarski [2, § 4] to Eq. (28), it is a straightforward matter to show that the θ vs. x solution profile admits *two* jumps and, moreover, to determine the amplitudes, locations, and speeds of said discontinuities. Here, as in Ref. [2], we define the amplitude of the jump discontinuity[5] in a function $\mathfrak{F} = \mathfrak{F}(x,t)$ across a wavefront $x = \Sigma(t)$ as

$$[\![\mathfrak{F}]\!] := \mathfrak{F}^- - \mathfrak{F}^+, \tag{35}$$

where $\mathfrak{F}^\mp := \lim_{x\to\Sigma(t)^\mp} \mathfrak{F}(x,t)$ are assumed to exist, and where a "+" superscript corresponds to the region into which Σ is advancing while a "−" superscript corresponds to the region behind Σ.

Introducing now the Rankine–Hugoniot conditions [16, 36] for our system of equations

$$[\![u]\!] = v_{1,2}[\![s]\!], \quad [\![\wp]\!] = \varrho_0 v_{1,2}[\![u]\!], \quad [\![u]\!] = \frac{b^2 v_{1,2}[\![\theta]\!]}{v_{1,2}^2 - b^2}, \quad [\![q]\!] = \frac{K_0\vartheta_0[\![\theta]\!]}{\tau_0 v_{1,2}}, \tag{36}$$

and noting that Eq. (15)$_3$ implies $[\![\wp]\!] = \wp_0[\![s]\!] + \wp_0[\![\theta]\!]$, we are able to express the jumps in the remaining field variables in terms of $[\![\theta]\!]$. Omitting the details, the following shock amplitude expressions are readily derived.

Across $\Sigma_1(t) = v_1 t$:

$$\begin{aligned} [\![\theta]\!] = \mathcal{A}_1 \exp(-\sigma_1 x), \quad [\![u]\!] &= \left(\frac{b^2 v_1 \mathcal{A}_1}{v_1^2 - b^2}\right)\exp(-\sigma_1 x), \\ [\![q]\!] &= \left(\frac{K_0\vartheta_0\mathcal{A}_1}{\tau_0 v_1}\right)\exp(-\sigma_1 x), \\ [\![s]\!] = \left(\frac{b^2\mathcal{A}_1}{v_1^2 - b^2}\right)\exp(-\sigma_1 x), \quad [\![\wp]\!] &= \wp_0\left(\frac{v_1^2\mathcal{A}_1}{v_1^2 - b^2}\right)\exp(-\sigma_1 x). \end{aligned} \tag{37}$$

Across $\Sigma_2(t) = v_2 t$:

$$\begin{aligned} [\![\theta]\!] = \mathcal{A}_2 \exp(-\sigma_2 x), \quad [\![u]\!] &= \left(\frac{b^2 v_2 \mathcal{A}_2}{v_2^2 - b^2}\right)\exp(-\sigma_2 x), \\ [\![q]\!] &= \left(\frac{K_0\vartheta_0\mathcal{A}_2}{\tau_0 v_2}\right)\exp(-\sigma_2 x), \\ [\![s]\!] = \left(\frac{b^2\mathcal{A}_2}{v_2^2 - b^2}\right)\exp(-\sigma_2 x), \quad [\![\wp]\!] &= \wp_0\left(\frac{v_2^2\mathcal{A}_2}{v_2^2 - b^2}\right)\exp(-\sigma_2 x). \end{aligned} \tag{38}$$

Here, we observe that

$$0 < v_1 < b < c_0 < v_2 < \infty \qquad (\tau_0 > 0), \tag{39}$$

i.e., $v_{1,2} \neq b$ since $\tau_0 > 0$ has been assumed, where for clarity we note that

$$\lim_{\tau_0\to 0} v_r = \begin{cases} b, & r = 1, \\ \infty, & r = 2, \end{cases} \qquad \lim_{\tau_0\to\infty} v_r = \begin{cases} 0, & r = 1, \\ c_0, & r = 2. \end{cases} \tag{40}$$

These wave speed results make clear the fact that $v_{1,2}$ are the acoustic (i.e., mechanical) and thermal (i.e., second-sound) wave speeds, respectively, where the blow-up of the latter in the limit $\tau_0 \to 0$ highlights the PHC.

In closing this subsection, we find it noteworthy that $v_{1,2}$ admit the following small-τ_0 approximations:

$$v_1 \approx b\left[1 - \frac{b^2(\gamma-1)\tau_0}{2\gamma\kappa}\right], \quad v_2 \approx \sqrt{\frac{\gamma\kappa}{\tau_0}}\left[1 + \frac{b^2(\gamma-1)\tau_0}{2\gamma\kappa}\right] \quad (\tau_0 \ll \kappa/b^2). \tag{41}$$

[5]It should be noted that the authors of Ref. [2] denote (what is written here as) $[\![\mathfrak{F}]\!]$ using the notation $\mathcal{S}\mathfrak{F}$, which is read "the *saltus* of $\mathfrak{F}$".

Unfortunately, however, *neither* is valid for air under normal conditions (i.e., 1 atm and 300 K), for which, based on Ref. [19, Eq. (18b)], $\tau_0 \approx 773$ picosec.

3.2. **Numerical results.** In Figure 1 we have plotted Q vs. x, where we have set $Q := \kappa q/(K_0\vartheta_0 u_0)$ for convenience. The profiles shown were computed for the case of air, under normal conditions, using Tzou's Riemann sum inversion formula [49, § 2.5.1], namely,

$$\mathfrak{F}(x,t) \approx \frac{e^{4.7}}{t}\left[\tfrac{1}{2}\overline{\mathfrak{F}}(x, 4.7/t) + \mathrm{Re}\left(\sum_{n=1}^{N}(-1)^n\,\overline{\mathfrak{F}}(x, (4.7+\mathrm{i}n\pi)/t)\right)\right] \quad (t>0), \quad (42)$$

to numerically invert Eq. (27). Here, $N(\gg 1)$ is an integer and $\mathrm{Re}(\cdot)$ denotes the real part of a complex quantity.

From Figure 1 it is clear that the MC case exhibits two shock fronts, $x = \Sigma_{1,2}$, while the Fourier case has only one, the speed of which, b, lies between $v_1 < v_2$. More interesting, however, is the following: (a) the MC-based Q profile is actually *increasing* on the interval between the two shock fronts; (b) the maximum value of the Fourier-based case exceeds that of its MC-based counterpart; and (c) the flux, in both cases, points *away* from the acoustic shock-front. Figure 1 also illustrates the highly transient nature of thermal wave phenomena in the present problem.

Remark 2. Under the scalings $\mathsf{U} \mapsto u/u_0$, $\Theta \mapsto \theta(b/u_0)$, $\mathsf{Q} = q/(u_0\varrho_0 c_v)$, $\mathsf{X} \mapsto x(\gamma\kappa/b)^{-1}$, and $\mathsf{T} \mapsto t(\gamma\kappa/b^2)^{-1}$, the inverse of the $\tau_0 := 0$ special case of $\overline{\mathsf{Q}}$ can be read directly from that of $\bar{\sigma}$ in Ref. [23] (i.e., Ref. [23, Eq. (9)]), where we observe that the quantity $(\gamma-1)$ here plays the role of ε, the thermoelastic coupling constant, in Ref. [23].

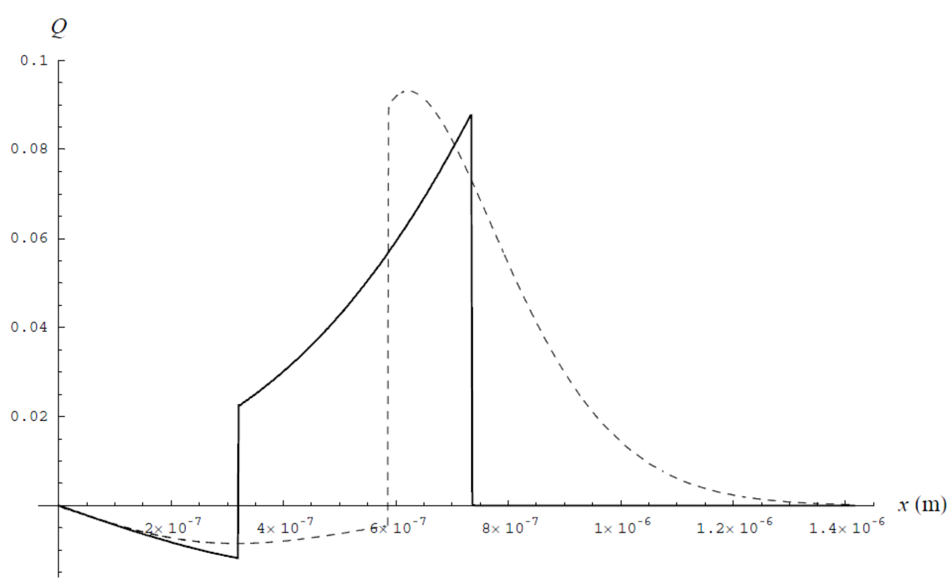

FIGURE 1. Q vs. x profile at time $t = 2.0$ nanosec, generated using Eq. (42) with $N = 20000$, for air at 1 atm and 300 K, where $\gamma = 1.4$, $\kappa \approx 2.22 \times 10^{-5}\,\mathrm{m^2/sec}$ [38], $b \approx 293.4$ m/sec, and $c_0 \approx 347.2$ m/sec; broken: $\tau_0 := 0$ (Fourier's law); solid: $\tau_0 \approx 773$ picosec (MC law).

4. **Weakly-nonlinear theory.** While they are exact, the equations of our governing system are, mathematically speaking, extremely complicated. Thus, so that further progress might be achieved, we now invoke the assumptions and arguments of weakly-nonlinear acoustics. Under this paradigm, we seek to synthesize "small, but finite-amplitude" (i.e., approximate) versions of our governing equations into a *single*, weakly-nonlinear equation of motion, where by weakly-nonlinear we mean a nonlinear PDE from which nonlinear terms of quadratic and higher order in the Mach number have been neglected.

First, however, we must recast our system into a more useful form. To this end, we observe that $\mathbf{u} = (u(x,t), 0, 0)$ implies $\nabla \times \mathbf{u} = 0$; therefore, $u = \phi_x$, where $\phi = \phi(x,t)$ denotes the velocity potential. Next, we introduce the following additional dimensionless quantities:

$$u^\diamond = u/V, \qquad \phi^\diamond = \phi/(LV), \qquad \mathfrak{e} = c_p^{-1}(\eta - \eta_0), \qquad P = \wp/\wp_0,$$
$$q^\diamond = q/q_{\text{mag}}, \qquad x^\diamond = x/L, \qquad t^\diamond = t(c_0/L), \tag{43}$$

where the positive constants L, V, and q_{mag} denote a characteristic length, speed, and value of $|\mathbf{q}|$, respectively. Now fully non-dimensionalized, Eqs. (4) and (10)–(13) can be written as

$$P = (1+s)^\gamma \exp(\gamma \mathfrak{e}), \tag{44}$$

$$s_t + \epsilon[\phi_x s_x + (1+s)\phi_{xx}] = 0, \tag{45}$$

$$\epsilon\gamma(1+s)\partial_x[\phi_t + \tfrac{1}{2}\epsilon(\phi_x)^2] = -P_x, \tag{46}$$

$$\tilde{K}(1+s)(1+\theta)(\mathfrak{e}_t + \epsilon\phi_x \mathfrak{e}_x) = -\tilde{\kappa} q_x, \tag{47}$$

$$q + \lambda_0(q_t + \epsilon\phi_x q_x) = -\tilde{K}\theta_x, \tag{48}$$

respectively. Here, $\epsilon = V/c_0$ and $\lambda_0 = \tau_0 c_0/L$ denote the Mach and the Knudsen numbers, respectively; $\tilde{K} = q_{\text{mag}}^{-1} K_0 \vartheta_0/L$ and $\tilde{\kappa} := c_0^{-1}\kappa/L$ denote the dimensionless thermal conductivity and the dimensionless thermal diffusivity, respectively; all diamond superscripts have been, and shall remain, suppressed; and for convenience we have introduced the notation $\partial_\zeta := \partial/\partial\zeta$.

In the remainder of this section, we shall proceed under the assumptions of the weakly-nonlinear approximation; specifically, that $\epsilon \ll 1$, $|s| = \mathcal{O}(\epsilon)$, $|\theta| = \mathcal{O}(\epsilon)$, $\lambda_0 = \mathcal{O}(\epsilon)$, $\tilde{\kappa} = \mathcal{O}(\epsilon)$, and $|\mathfrak{e}| = \mathcal{O}(\epsilon^2)$. Additionally, we note for future reference the fact that

$$s = -\epsilon\phi_t + \mathcal{O}(\epsilon^2), \tag{49}$$

a result which is readily derived from Bernoulli's theorem,

4.1. **Bi-directional equation of motion: Derivation.** As a first step, we expand Eq. (44), which we are free to do since $|s| \ll 1$ and $|\mathfrak{e}| \ll 1$ have been assumed, to find that

$$P = 1 + \gamma[s + \tfrac{1}{2}(\gamma - 1)s^2 + \mathfrak{e} + \cdots]. \tag{50}$$

Next, we replace P in Eq. (46) with the right-hand side of Eq. (50) and then carry out the indicated differentiation with respect to x on the right-hand side of the former. Now dividing both sides by $1 + s$ and then expanding $(1+s)^{-1}$ in a binomial series, recalling the weakly-nonlinear assumption $|s| = \mathcal{O}(\epsilon)$, our momentum equation becomes, after simplifying and neglecting terms of $\mathcal{O}(\epsilon^2)$,

$$\partial_x\{\phi_t + \tfrac{1}{2}\epsilon(\phi_x)^2 + \epsilon^{-1}[s + \tfrac{1}{2}(\gamma - 2)s^2 + \mathfrak{e}]\} = 0. \tag{51}$$

Integrating now with respect to x and then applying ∂_t to both sides, Eq. (51) becomes

$$\phi_{tt} + \tfrac{1}{2}\epsilon\partial_t(\phi_x)^2 + \epsilon^{-1}[1+(\gamma-2)s]s_t + \epsilon^{-1}\mathfrak{e}_t = 0, \tag{52}$$

which after using Eq. (45) to eliminate s_t becomes, in turn,

$$\phi_{tt} + \tfrac{1}{2}\epsilon\partial_t(\phi_x)^2 - [1+(\gamma-2)s][\phi_x s_x + (1+s)\phi_{xx}] = -\epsilon^{-1}\mathfrak{e}_t. \tag{53}$$

Turning our attention now to Eqs. (47) and (48), we observe that both are used in linearized form under the weakly-nonlinear paradigm; i.e., the (dimensionless) balance law expressing the rate of (specific) entropy production reduces to [15, 29]

$$\tilde{K}\mathfrak{e}_t = -\tilde{\kappa}q_x, \tag{54}$$

which we note holds for sound fields far removed from solid boundaries, and the constitutive relation for the (dimensionless) heat flux once again[6] assumes the form

$$(1+\lambda_0\partial_t)q = -\tilde{K}\theta_x. \tag{55}$$

On eliminating q between the former and latter, our linearized entropy equation becomes

$$(1+\lambda_0\partial_t)\mathfrak{e}_t = \tilde{\kappa}\theta_{xx}, \tag{56}$$

which we can immediately recast as

$$(1+\lambda_0\partial_t)\mathfrak{e}_t = -\epsilon\tilde{\kappa}(\gamma-1)\phi_{txx}, \tag{57}$$

via Eq. (49) and the approximation [9, p. 46]

$$\theta \approx -\epsilon(\gamma-1)\phi_t. \tag{58}$$

Returning to Eq. (53) and applying the relaxation operator $(1+\lambda_0\partial_t)$, followed by the use of Eq. (57) to eliminate $\mathfrak{e}_t$, our momentum equation becomes, after rearranging terms and simplifying,

$$\begin{aligned}(1+\lambda_0\partial_t)\{\phi_{tt} + \tfrac{1}{2}\epsilon\partial_t(\phi_x)^2 - [1+(\gamma-2)s][\phi_x s_x + (1+s)\phi_{xx}]\}\\ = (\mathrm{Re}_\theta)^{-1}\phi_{txx}.\end{aligned} \tag{59}$$

Here, Re_θ, what we term the *thermal* Reynolds number, is given by

$$\mathrm{Re}_\theta := [\tilde{\kappa}(\gamma-1)]^{-1} = c_0L/\delta_\theta, \tag{60}$$

where $\delta_\theta := \kappa(\gamma-1)$ denotes the inviscid special case of the diffusivity of sound [46]. Again making use of Eq. (49), but now to eliminate s and s_x, Eq. (59) assumes the form

$$(1+\lambda_0\partial_t)\{\phi_{tt} - \phi_{xx} + \epsilon[\partial_t(\phi_x)^2 + (\gamma-1)\phi_t\phi_{xx} + \mathcal{O}(\epsilon)]\} = (\mathrm{Re}_\theta)^{-1}\phi_{txx}. \tag{61}$$

Finally, on applying $(1+\lambda_0\partial_t)$ to each term on the left-hand side of Eq. (61), and thereafter neglecting terms of $\mathcal{O}(\epsilon^2)$ and simplifying, we obtain the following as our (bi-directional) weakly-nonlinear equation of motion:

$$\lambda_0\phi_{ttt} + \phi_{tt} - [1 - 2\epsilon(\beta-1)\phi_t]\phi_{xx} - \sigma\phi_{txx} + \epsilon\partial_t(\phi_x)^2 = 0, \tag{62}$$

a PDE which can also be expressed as

$$\phi_{tt} - [1-2\epsilon(\beta-1)\phi_t]\phi_{xx} - (\mathrm{Re}_\theta)^{-1}\phi_{txx} + \epsilon\partial_t(\phi_x)^2 = -\lambda_0\partial_t(\phi_{tt} - \phi_{xx}). \tag{63}$$

Here, for convenience, we have introduced $\beta(>1)$, known as the coefficient of nonlinearity [1, 29], which in the case of a perfect gas is given by $\beta = (\gamma+1)/2$, and we have set $\sigma := \lambda_0 + (\mathrm{Re}_\theta)^{-1}$. It is noteworthy that the linearized version of Eq. (62)

[6]Note that Eq. (55) is just a dimensionless version of Eq. (16).

is equivalent to models which have arisen in a number of acoustics-related fields; see, e.g., Refs. [25, 31, 42, 46].

Remark 3. If we let $\lambda_0 \to 0$ (i.e., $\tau_0 \to 0$), then it is a simple matter to show that Eq. (63) reduces to

$$\phi_{tt} - [1 - 2\epsilon(\beta - 1)\phi_t]\phi_{xx} - (\mathrm{Re}_\theta)^{-1}\phi_{txx} + \epsilon\partial_t(\phi_x)^2 = 0. \tag{64}$$

This PDE, which of course is based on Fourier's law, is the special case of the 1D Blackstock–Lesser–Seebass–Crighton (BLSC) equation corresponding to an inviscid, thermally conducting gas; see, e.g., Ref. [22] and those therein.

4.2. **Traveling wave analysis.** Confining our attention to only right-running waves, which we do without loss of generality since Eq. (62) is invariant under the transformation $x \mapsto -x$, we introduce the ansatzes $u(x,t) = f(\xi)$ and $\phi(x,t) = F(\xi)$, where $\xi := x - ct$ is the wave variable and the constant $c(> 0)$ represents the speed of the traveling wave. Making these substitutions and then integrating the resulting ODE with respect to ξ, subject to the asymptotic condition $f \to 0$ as $\xi \to \infty$, Eq. (62) is reduced to the quadratic Bernoulli equation

$$-c(c^2\lambda_0 - \sigma)f' + (c^2 - 1)f - c\epsilon\beta f^2 = 0, \tag{65}$$

where a prime denotes $\mathrm{d}/\mathrm{d}\xi$, and we note that $f = F'$.

On imposing and enforcing the second of our asymptotic conditions, i.e., $f \to 1$ as $\xi \to -\infty$, the speed of our traveling waveform is found to be *identical* to that admitted by the BLSC equation (i.e., Eq. (64)), namely,

$$c = \tfrac{1}{2}\epsilon\beta + \sqrt{1 + \tfrac{1}{4}\epsilon^2\beta^2}\,, \tag{66}$$

where we note that $c > 1$ follows immediately from the fact that

$$c = 1 + \frac{1}{2}\epsilon\beta + \frac{1}{8}\epsilon^2\beta^2 + \cdots \quad (\epsilon \ll 1). \tag{67}$$

Since c, as given in Eq. (66), is the positive root of $c^2 - 1 = \epsilon\beta c$, Eq. (65) can be further reduced to

$$-(c^2\lambda_0 - \sigma)f' + \epsilon\beta(f - f^2) = 0, \tag{68}$$

which is easily integrated and yields the *Taylor shock* solution

$$f(\xi) = \tfrac{1}{2}\{1 - \tanh[2\xi/l(\lambda_0)]\} \qquad (\lambda_0 < \lambda_0^*). \tag{69}$$

Here, $l(> 0)$, the shock thickness [38, p. 591], and λ_0^*, a critical value of λ_0, are given by

$$l(\lambda_0) = \left(1 - \frac{\lambda_0}{\lambda_0^*}\right)\ell_{\mathrm{BLSC}} \qquad \text{and} \qquad \lambda_0^* := \frac{1}{c\epsilon\beta\mathrm{Re}_\theta}, \tag{70}$$

where $\ell_{\mathrm{BLSC}} = 4(\epsilon\beta\mathrm{Re}_\theta)^{-1}$ denotes the shock thickness corresponding to Eq. (64) and the restriction $\lambda_0 < \lambda_0^*$ has been imposed to ensure that f satisfies the imposed asymptotic conditions.

Remark 4. It is noteworthy that the requirement $0 < l < \ell_{\mathrm{BLSC}}$ is equivalent to

$$0 < c\lambda_0 < \tfrac{1}{4}\ell_{\mathrm{BLSC}}. \tag{71}$$

4.3. **Unidirectional approximation: The hyperbolic Burgers equation.** Let us now divide Eq. (62) by $[1 - 2\epsilon(\beta - 1)\phi_t]$, which can never be zero, expand each occurrence of the reciprocal of this quantity in a binomial series, based on $\epsilon \ll 1$, and then, as before, neglect all terms of $\mathcal{O}(\epsilon^2)$. Our equation of motion then becomes, after simplifying,

$$\lambda_0\phi_{ttt} + \phi_{tt} - \phi_{xx} - \sigma\phi_{txx} + \epsilon\partial_t[(\phi_x)^2 + (\beta - 1)(\phi_t)^2] = 0. \tag{72}$$

In the limit $\lambda_0 \to 0$, this PDE reduces to the special case of Kuznetsov's equation corresponding to an inviscid, thermally conducting fluid; again, see Ref. [22] and those therein.

Since we are only concerned with right-running waves (i.e., $c > 0$), we can replace[7], based on the approximation $\phi_x \simeq -\phi_t$, the difference $\phi_{tt} - \phi_{xx}$ with $2\partial_t(\partial_t + \partial_x)\phi$, and in doing so Eq. (72) becomes

$$\lambda_0\phi_{ttt} + 2\partial_t(\partial_t + \partial_x)\phi - \sigma\phi_{txx} + \epsilon\partial_t[(\phi_x)^2 + (\beta - 1)(\phi_t)^2] = 0. \tag{73}$$

If we now replace the "small" term $(\phi_t)^2$ with $(\phi_x)^2$, again based on the approximation $\phi_x \simeq -\phi_t$, integrate the result with respect to t, and then differentiate with respect to x, Eq. (73) is reduced to

$$\tfrac{1}{2}(\lambda_0 u_{tt} - \sigma u_{xx}) + u_t + (1 + \epsilon\beta u)u_x = 0, \tag{74}$$

where we have also made use of the relation $u = \phi_x$. Equation (74), which has come to be known as the hyperbolic Burgers equation (HBE), arises in a number of diverse fields, the earliest and best known of which being traffic flow modeling under kinematic-wave theory; see, e.g., Refs. [11, 21] and the those therein.

It is fortunate, indeed, that the findings presented by Christov and Jordan [11], all of which were derived in the traffic flow context, can be directly applied to the present study. This is readily accomplished by first recasting Eq. (74) as

$$\begin{pmatrix} u \\ j \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ a_0^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ j \end{pmatrix}_x = \frac{2}{\lambda_0}\begin{pmatrix} 0 \\ u(1 + \frac{1}{2}\epsilon\beta u) - j \end{pmatrix}, \tag{75}$$

which we note is a strictly hyperbolic, *semilinear* system with characteristics defined by $\mathrm{d}x/\mathrm{d}t = \pm a_0$ (see, e.g., Dafermos [16]). Then, use is made of the fact that the dependent variables and parameters of this system are related to their kinematic counterparts in Ref. [11, Eq. (4)] as follows:

$$\rho \mapsto -u, \quad q \mapsto -j; \quad \tau_0 = \tfrac{1}{2}\lambda_0, \quad \nu = \tfrac{1}{2}\sigma, \quad \rho_{\mathrm{s}} = \frac{2}{\epsilon\beta}, \quad v_{\mathrm{m}} = 1, \quad c_0 = a_0. \tag{76}$$

Here, the kinematic quantities, which are those appearing on the left-hand side of each replacement listed in Eq. (76), are defined in Ref. [11]; j has been introduced to play the role of a flux; $a_0 = \sqrt{\sigma/\lambda_0}$ denotes the characteristic speed; and to simplify the analysis, $\lambda_0 > 0$ (i.e., $\tau_0 > 0$) is henceforth assumed.

In Figure 2 we compare v_0, the *dimensional* version of a_0, where the former is given by

$$v_0 = c_0\sqrt{1 + c_0^{-2}\kappa(\gamma - 1)/\tau_0}, \tag{77}$$

with $v_{1,2}$, the acoustic and thermal wave speeds, respectively, from section 3, for the case of air under normal conditions. The curves shown clearly indicate that v_0 is the weakly-nonlinear approximation to the speed of the *thermal* wave, not the acoustic one, and that $v_1 \to b$ (from below) as $\tau_0 \to 0$, a consequence of the fact that v_1 assumes its maximum value (i.e., b) under Fourier's law. Figure 2 also makes

[7]See, e.g., Crighton's [14, p. 16] reduction of Eq. (64) to Burgers equation.

clear that for $\tau_0 \gtrsim 773$ picosec, where we recall that $\tau_0 \approx 773$ picosec is the value computed in the case of air using Ref. [19, Eq. (18b)], the v_0 and v_2 curves show very good/excellent agreement, with both tending to c_0 (from above) as $\tau_0 \to \infty$.

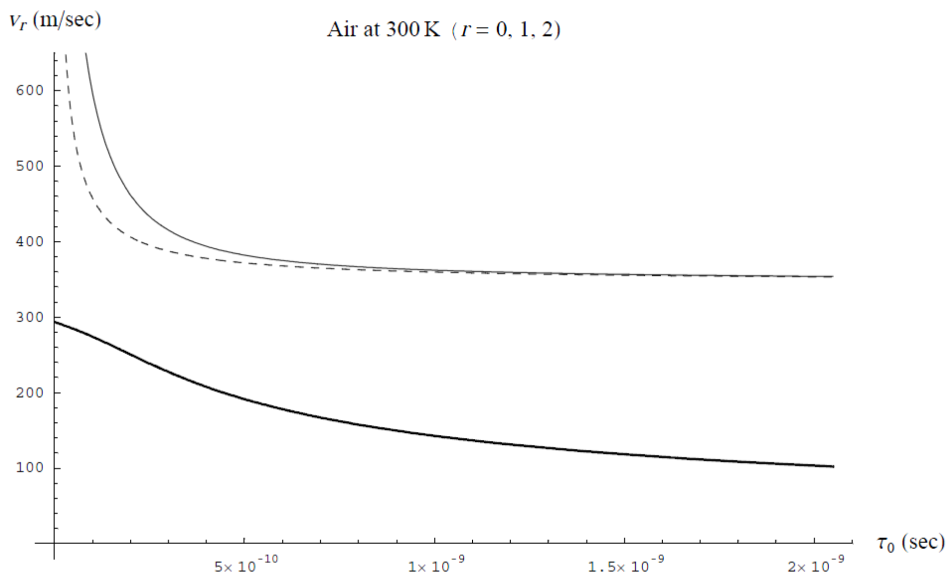

FIGURE 2. Wave speed profiles as functions of τ_0 in the case of air at 1 atm and 300 K, for which $\gamma = 1.4$, $\kappa \approx 2.22 \times 10^{-5}$ m^2/sec [38], $b \approx 293.4$ m/sec, and $c_0 \approx 347.2$ m/sec; broken: v_0 vs. τ_0; bold-solid: v_1 vs. τ_0; and thin-solid: v_2 vs. τ_0.

Remark 5. In studies of second-sound in solids, and applications based on the models thereof, one often encounters PDEs quite similar to Eq. (74). For example, in her work on second-sound in rigid conductors with memory, Carillo [3, 4] has derived and analyzed the nonlinear wave equation

$$u_{tt} = k_0(u_{xx} + 2uu_x), \tag{78}$$

which Jordan [21] has shown can also be derived under what he terms "inertial-Type-II" theory, while Straughan [43] has obtained acceleration results for the model

$$u_{tt} + \alpha u_t = D\alpha u_{xx} + \alpha k f(u), \tag{79}$$

which is a hyperbolic generalization of the class of PDEs known as (1D) reaction-diffusion equations. In Eqs. (78) and (79), $k_0(> 0)$, $D(> 0)$, $\alpha(> 0)$, and $k(\geq 0)$ are constants.

4.4. **Singular surface results: Thermoacoustic shocks.**

4.4.1. *Shocks in the velocity field.* We now turn our attention to understanding the impact of thermal relaxation on thermoacoustic shocks; in particular, the evolution of the shock amplitude, which in the case of the velocity field is once again denoted by $[\![u]\!]$, across the (right-running) shock-front $x = \Sigma(t)$. Here, $\Sigma(t) = a_0t + x_0$, where the constant x_0 denotes the initial location of Σ on the x-axis; recall section 3.1.

Using the machinery of singular surface theory [34, 36, 45], it is readily established that $\mathcal{S}(t)$, where we have set $\mathcal{S}(t) = \frac{1}{2}\epsilon\beta[\![u]\!]$ for convenience, like its kinematic counterpart $S(t)$ in Ref. [11], satisfies a Bernoulli equation, specifically,

$$a_0\lambda_0 \frac{\mathcal{D}\mathcal{S}}{\mathcal{D}t} = -\alpha^\bullet \mathcal{S} + \mathcal{S}^2, \tag{80}$$

which integrates to

$$\mathcal{S}(t) = \begin{cases} \dfrac{\alpha^\bullet}{1 - \left(1 - \frac{\alpha^\bullet}{\mathcal{S}(0)}\right)\exp\left[\left(\frac{\alpha^\bullet}{a_0\lambda_0}\right)t\right]}, & \alpha^\bullet \neq 0, \\ \dfrac{a_0\lambda_0\mathcal{S}(0)}{a_0\lambda_0 - \mathcal{S}(0)t}, & \alpha^\bullet = 0, \end{cases} \qquad \mathcal{S}(0) \neq 0. \tag{81}$$

Here, $\mathcal{D}/\mathcal{D}t$, the 1D displacement derivative, gives the time-rate-of-change measured by an observer traveling with Σ, and $\alpha^\bullet$, the critical shock amplitude, is given by

$$\alpha^\bullet = a_0 - (1 + \epsilon\beta u^+) = -\left[(1 + \epsilon\beta u^+) - \sqrt{1 + \frac{1}{\lambda_0}\left(\frac{1}{\mathrm{Re}_\theta}\right)}\right], \tag{82}$$

where u^+, the value of u immediately ahead of Σ, has been assumed constant.

At this stage it is instructive to briefly shift our analysis to the phase plane and examine the stability of the (two) equilibria of Eq. (80), which we denote using a superposed hat (i.e., $\hat{\mathcal{S}}$). Omitting the details, but referring the reader to any of the many excellent texts which treat qualitative methods for ODEs, the following can be established without difficulty:

(i) If $u^+ > 0$ and $\lambda_0 < \lambda_0^\bullet$, then $\alpha^\bullet > 0$; therefore $\hat{\mathcal{S}} = \{0, \alpha^\bullet\}$ are stable and unstable, respectively.

(ii) If $u^+ > 0$ and $\lambda_0 = \lambda_0^\bullet$, then $\alpha^\bullet = 0$; therefore the equilibria has coalesced at $\hat{\mathcal{S}} = 0$, which is now both a double zero of the quadratic on the right-hand side of Eq. (80) *and* a left-semi-stable bifurcation point.

(iii) If $u^+ > 0$ and $\lambda_0 > \lambda_0^\bullet$, then $\alpha^\bullet < 0$; therefore $\hat{\mathcal{S}} = \{0, \alpha^\bullet\}$ are unstable and stable, respectively.

(iv) If $u^+ \leq 0$, then $\alpha^\bullet > 0$; therefore $\hat{\mathcal{S}} = \{0, \alpha^\bullet\}$ are stable and unstable, respectively.

Here, the bifurcation value of Eq. (80) defines the critical λ_0-value

$$\lambda_0^\bullet := [2\epsilon\beta u^+(1 + \tfrac{1}{2}\epsilon\beta u^+)\mathrm{Re}_\theta]^{-1}, \tag{83}$$

i.e., $\alpha^\bullet = 0$ for $\lambda_0 = \lambda_0^\bullet$, where we observe that a necessary condition for $\lambda_0^\bullet$ to be physically well-defined is $u^+ > 0$.

Regarding the evolution of $\mathcal{S}(t)$, Eq. (81) indicates that, from the mathematical standpoint, this can occur in any one of the following nine ways:

(I) If $\alpha^\bullet > 0$ and $\mathcal{S}(0) < 0$, then $\mathcal{S}(t) \to 0$ (from below) as $t \to \infty$.
(II) If $\alpha^\bullet > 0$ and $\mathcal{S}(0) \in (0, \alpha^\bullet)$, then $\mathcal{S}(t) \to 0$ (from above) as $t \to \infty$.
(III) If $\alpha^\bullet > 0$ and $\mathcal{S}(0) > \alpha^\bullet$, then $\mathcal{S}(t) \to \infty$ as $t \to t_\infty$.
(IV) If $\alpha^\bullet = 0$ and $\mathcal{S}(0) < 0$, then $\mathcal{S}(t) \to 0$ (from below) as $t \to \infty$.
(V) If $\alpha^\bullet = 0$ and $\mathcal{S}(0) > 0$, then $\mathcal{S}(t) \to \infty$ as $t \to t_\infty$.
(VI) If $\alpha^\bullet < 0$ and $\mathcal{S}(0) < \alpha^\bullet$, then $\mathcal{S}(t) \to \alpha^\bullet$ (from below) as $t \to \infty$.
(VII) If $\alpha^\bullet < 0$ and $\mathcal{S}(0) \in (\alpha^\bullet, 0)$, then $\mathcal{S}(t) \to \alpha^\bullet$ (from above) as $t \to \infty$.
(VIII) If $\alpha^\bullet < 0$ and $\mathcal{S}(0) > 0$, then $\mathcal{S}(t) \to \infty$ as $t \to t_\infty$.
(IX) If $\alpha^\bullet \neq 0$ and $\mathcal{S}(0) = \alpha^\bullet$, then $\mathcal{S}(t) = \alpha^\bullet$ for all $t \geq 0$.

Here, t_∞, the time at which finite-time blow-up occurs, is given by

$$t_\infty = a_0\lambda_0 \begin{cases} (\alpha^\bullet)^{-1}\ln\left(\dfrac{\mathcal{S}(0)}{\mathcal{S}(0)-\alpha^\bullet}\right), & \alpha^\bullet \neq 0,\\ 1/\mathcal{S}(0), & \alpha^\bullet = 0, \end{cases} \tag{84}$$

where we observe that $t_\infty > 0$ holds *only* under Cases (III), (V), and (VIII), and we recall the assumption $\mathcal{S}(0) \neq 0$.

4.4.2. *Shocks in the density and temperature fields.* If, instead, we had replaced $(\phi_x)^2$ with $(\phi_t)^2$, again based on $\phi_x \simeq -\phi_t$, followed by use of the approximation $s \approx -\epsilon\phi_t$ (recall Eq. (49)), then Eq. (73) would have reduced to

$$\tfrac{1}{2}(\lambda_0 s_{tt} - \sigma s_{xx}) + s_x + s_t = \tfrac{1}{2}\beta\partial_t(s^2). \tag{85}$$

While certainly not intractable, this PDE can be put into an even more useful form by replacing the operator ∂_t on the right-hand side with $-\partial_x$; the result of this approximation is

$$\tfrac{1}{2}(\lambda_0 s_{tt} - \sigma s_{xx}) + s_t + (1+\beta s)s_x = 0, \tag{86}$$

which of course is the HBE expressed in terms of the condensation.

With Eq. (86) in hand, it is a relatively simple matter to show that, in terms of the condensation/density field, the evolution of the shock amplitude is, just as in the case of the velocity field, described by the solutions of a Bernoulli equation. On setting $\mathcal{R}(t) := \frac{1}{2}\beta[\![s]\!]$, we have

$$\mathcal{R}(t) = \begin{cases} \dfrac{\alpha^\star}{1-\left(1-\frac{\alpha^\star}{\mathcal{R}(0)}\right)\exp\left[\left(\frac{\alpha^\star}{a_0\lambda_0}\right)t\right]}, & \alpha^\star \neq 0,\\ \dfrac{a_0\lambda_0\mathcal{R}(0)}{a_0\lambda_0 - \mathcal{R}(0)t}, & \alpha^\star = 0, \end{cases} \qquad \mathcal{R}(0) \neq 0. \tag{87}$$

Here, $\alpha^\star$, the critical shock amplitude, is given by

$$\alpha^\star = a_0 - (1+\beta s^+) = -\left[(1+\beta s^+) - \sqrt{1+\frac{1}{\lambda_0}\left(\frac{1}{\mathrm{Re}_\theta}\right)}\right]; \tag{88}$$

s^+, the value of s immediately ahead of Σ, is assumed constant; and we observe that $\alpha^\star = 0$, the bifurcation value in this case, defines the second critical λ_0-value

$$\lambda_0^\star := [2\beta s^+(1+\tfrac{1}{2}\beta s^+)\mathrm{Re}_\theta]^{-1}, \tag{89}$$

where a necessary condition for $\lambda_0^\star$ to be physically well-defined is $s^+ > 0$.

Lastly, to determine $[\![\theta]\!]$, we simply take jumps of Eq. (58) and then make use of Eq. (87), recalling Eq. (49). After expressing γ in terms of β, it can be shown that

$$[\![\theta]\!] \approx 2(\beta-1)[\![s]\!] = 4(1-\beta^{-1})\mathcal{R}(t). \tag{90}$$

Remark 6. It would be of interest to perform a detailed comparison of the thermoviscous shock results obtained by Morro [34], who employed a “hidden variable” fluid model, with those derived here under the weakly-nonlinear approximation.

5. Closure.

- For $\tau_0 > 0$, the solution profiles of section 3 all admit two propagating shock-fronts, the slower one acoustic with speed v_1, the other thermal with speed v_2; see the example in Figure 1. In the limit $\tau_0 \to 0$, however, these profiles admit only an acoustic wavefront, which can be either a shock or an acceleration wave, depending on the variable in question, that propagates with speed b.
- In applying the weakly-nonlinear approximation to System (44)–(48), the resulting equation of motion describes *only* the thermal wave, i.e., the wave with (dimensional) characteristic speed v_0, where v_0 is such that (see Figure 2)

$$0 < v_1 < b < c_0 < v_0 < v_2 < \infty \qquad (\tau_0 > 0); \tag{91}$$

  compare these values with those derived by Straughan [45] for the speed of acceleration waves in Cattaneo–Christov gases.
- The traveling wave inequality $0 < l < \ell_{\mathrm{BLSC}}$ carries with it the following physical implication: Introducing thermal relaxation via the MC law *mitigates*, with respect to classical (i.e., Fourier-based) gas dynamics theory, the effects of thermally-induced dissipation in perfect gases.
- The Taylor shock described by Eq. (69) can "shock-up" in *two* different ways, namely, in the limit $\mathrm{Re}_\theta \to \infty$, as is also true in the case of the BLSC equation, *and* in the limit $\lambda_0 \to \lambda_0^*$ (from below), with the value of δ_θ remaining fixed.
- Under the HBE version of our equation of motion, a *necessary* mathematical condition for the shock amplitude $\mathcal{S}(t)$ to exhibit a bifurcation is $u^+ > 0$.
- The three cases of finite-time blow-up detailed in section 4.4.1 reflect a breakdown of our mathematical model (i.e., the HBE); specifically, in Case (III) the initial data is evidently too large while in Cases (V) and (VIII) the value of λ_0 is equal to or greater than, respectively, that of $\lambda_0^\bullet$ (see Eq. (83)).
- To examine the coupling between, and the structure of, 1D thermal and acoustic waves in a Cattaneo–Christov gas under the weakly-nonlinear approximation, one would have to, based on the results of section 3, begin by re-deriving the following system under the assumption that, instead of Fourier's law, the heat flux obeys Eq. (55):

$$\epsilon\{\gamma\phi_{tt} + \tfrac{1}{2}\epsilon(\gamma+1)\partial_t(\phi_x)^2 - [1 - \epsilon(\gamma-1)\phi_t]\phi_{xx}\} = -\theta_t, \tag{92}$$

$$\theta_t + \epsilon\phi_x\theta_x - \gamma\tilde{\kappa}\theta_{xx} = -\epsilon(\gamma-1)[1 - \epsilon(\gamma-1)\phi_t]\phi_{xx}. \tag{93}$$

- Another possible follow-on study involves setting aside the weakly-nonlinear approximation and performing an "exact" traveling wave analysis based on the (1D) system in section 2, thereby generalizing that of Christov et al. [12] to Cattaneo–Christov gases.

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